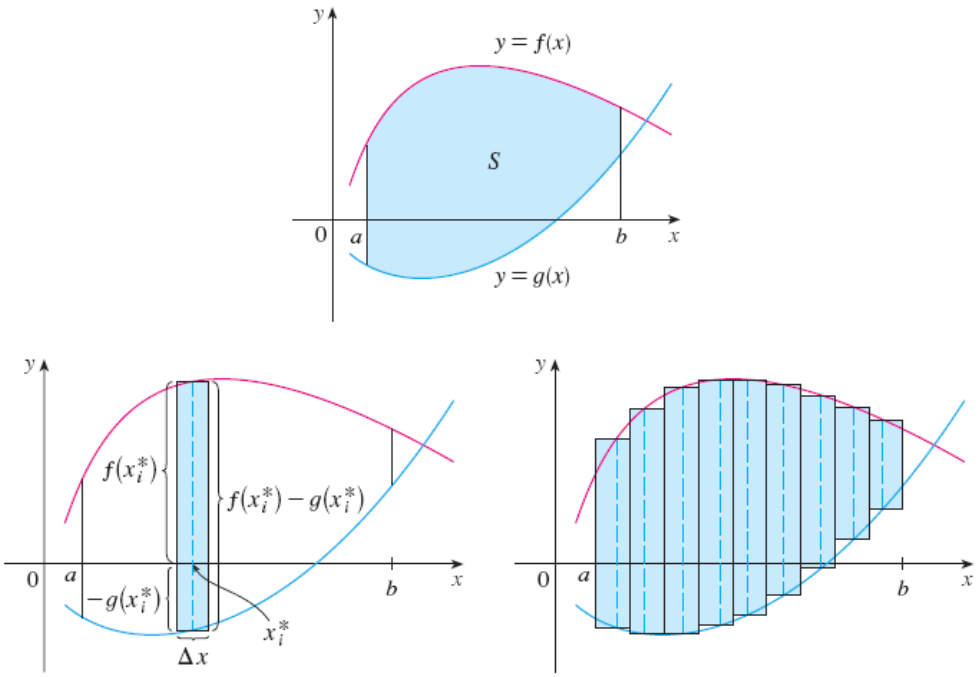
Chapter 5 Applications of Integration

Area Between Two Curves

In this section we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[\, a\, ,\, b\,]$.

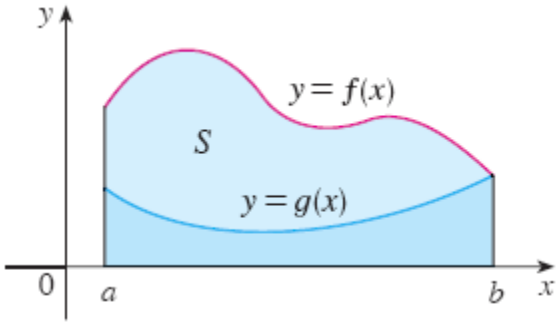


(a) Typical rectangle (b) Approximating rectangles

The area A of this region is

$$A = \int_a^b [f(x) - g(x)]dx$$

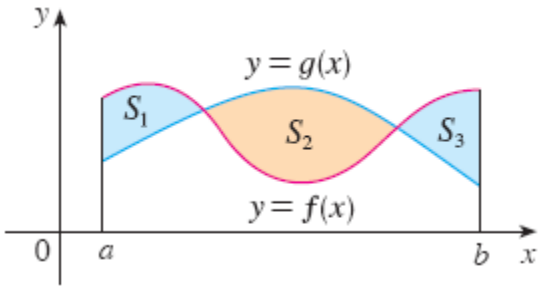
Note:



A = area under $f(x)$ – area under $g(x)$

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx$$



If we are asked to find the area where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, we have the following expression of A.

Area Between Curves: The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|dx$$

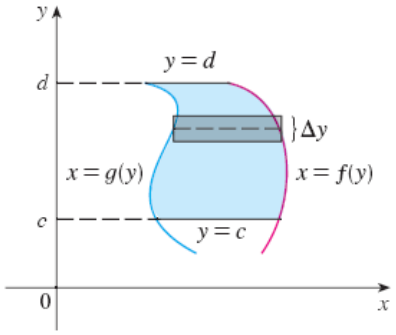
Note:

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) \text{ if } f(x) \geq g(x) \\ g(x) - f(x) \text{ if } g(x) \geq f(x) \end{cases}$$

Example 1: Find the area bounded by $y = x^2$, $y = x^3$, $x = 0$, and $x = 2$.

Example 2: Find the area enclosed by $y = |x|$, $y = x^2 - 2$.

Example 3: Find the area enclosed by $y = x$, $x + 2y = 0$, $2x + y = 3$.



Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then the area is

$$A = \int_c^d [f(y) - g(y)]dy$$

Example 4: Find the area enclosed by $4x + y^2 = 12$, $x = y$.